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Mathematical Minutiae: Differentiation as a Functor

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Unlike any other article in this journal, this one begins with a warning: Categories, beautiful and powerful as they may be, are not panacea and should be used with great prudence. This short note presents a fun, but silly use of categories.

7.1 The Chain Rule

In what follows, $\mathbb{R}$ denotes the set of real numbers. By $\tau_{\mathbb{R}}$ we mean the category whose objects are pairs (U, u) of open subsets $U \subseteq \mathbb{R}$ together with a point $u \in U$, and whose morphisms $(U, u) \to (U', u')$ are differentiable functions f preserving base points, in the sense that $f(u) = u'$. By $\mathscr{R}$ we mean the category whose unique object is $\mathbb{R}$, and whose morphisms are given by

$$\mathrm{Hom}_{\mathscr{R}}(\mathbb{R}, \mathbb{R}) = \{\phi_a : x \mapsto ax \mid a \in \mathbb{R}\};$$

the composition of ϕ_a and ϕ_b is defined to be ϕ_{ab}. We now claim that the assignation $D : \tau_{\mathbb{R}} \to \mathscr{R}$ given by

$$\begin{array}{rcl} (U,u) & \longmapsto & \mathbb{R} \\ (U,u) \xrightarrow{f} (U',u') & \longmapsto & \left.\dfrac{df}{dx}\right|_{x=u} \end{array}$$

is a functor.

Indeed, we need to check that, given a diagram of the form

$$(U, u) \xrightarrow{f} (U', u') \xrightarrow{g} (U'', u''),$$

the following relation holds:

$$D(g \circ f) = D(g) \circ D(f).$$

But this last expression can be rewritten as $(g \circ f)'(u) = g'(u')f'(u)$, which is exactly the chain rule at u! Moreover, to say that D preserves the identity is precisely to say that the derivative of $f(x) = x$ is 1, which is clearly true.

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7.2 Getting more serious

A rather more fruitful way to think about derivations in terms of functors is that of modern geometry. We pursue this with extreme economy, at the expense of using many undefined words. Let's think of smooth manifolds as ringed spaces, i.e., pairs $(M, \mathscr{O}_M)$ consisting of a topological space together with a sheaf of functions, such that $(M, \mathscr{O}_M)$ is locally isomorphic to $(\mathbb{R}^n, \mathscr{O}_{\text{sm}})$, the ringed space of $\mathbb{R}^n$ together with the sheaf of smooth functions on it. To every point of M we may attach a ring, that of the derivations from the stalk of the structure sheaf $\mathscr{O}_{\text{sm},m}$ to $\mathbb{R}$—this is a well-known gadget, the **tangent space** at m. Now, there is a way of compiling all these tangent spaces into the **tangent sheaf** on M, which is the dual to the better-known sheaf of differential forms $\Omega_{M/\mathbb{R}}$. And that these sheaves, like all sheaves, are functors of some sort, should please any rabid categorialist.