FEATURE

7

MATHEMATICAL MINUTIAE

# Irrational Numbers and the Euclidean Algorithm

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Remember in middle school when we first learned the difference between **rational** and **irrational** numbers? Informally, we were told that irrational numbers could not be represented as fractions of integers. But now we will see how a number theoretic algorithm based on the simple concept of division can yield fraction representations of irrational numbers.

First, we define the standard division algorithm in $\mathbb{Z}$, the set of integers.

**Definition 1** (Division Algorithm)**.** The **division algorithm** in $\mathbb{Z}$ states that for all $a, b \in \mathbb{Z}$, there exist $q, r \in \mathbb{Z}$ such that

$$a = bq + r, \qquad 0 \leq r < b.$$

This algorithm precisely matches our intuition about division in the integers. By recursively applying the division algorithm, we obtain the famous algorithm of Euclid:

**Definition 2** (Euclidean Algorithm)**.** The **Euclidean algorithm** in $\mathbb{Z}$ is a repeated division process, beginning with the division algorithm on two integers $a$ and $b$ and proceeding as follows:

$$\begin{aligned} a &= b \cdot q_1 + r_1, & 0 &\leq r_1 < b, \\ b &= r_1 \cdot q_2 + r_2, & 0 &\leq r_2 < r_1, \\ r_1 &= r_2 \cdot q_3 + r_3, & 0 &\leq r_3 < r_2, \\ &\vdots & &\vdots \\ r_{k-2} &= r_{k-1} \cdot q_k + r_k, & 0 &\leq r_k < r_{k-1}. \end{aligned}$$

The Euclidean algorithm stops when the remainder in the division algorithm is 0. In the above representation, $k$ is the number of steps in the algorithm, $r_{k-1}$ is the last non-zero remainder, and $r_k = 0$. (A proof that the Euclidean algorithm eventually stops for every pair of integers $a$ and $b$ is left as an exercise to the reader.) We introduce one last definition:

**Definition 3** (Fraction Sequence)**.** The **fraction sequence** $[a_1, a_2, a_3, \ldots, a_n]$ is equal to the following continued fraction expansion:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}$$

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It is a wonderful result in elementary number theory that if $a$ and $b$ are integers, then $\frac{a}{b} = [q_1, q_2, q_3, \ldots, q_k]$, where $q_1, q_2, q_3, \ldots, q_k$ are the sequence of quotients from the Euclidean algorithm on $a$ and $b$. Furthermore, the division and Euclidean algorithms can be extended for irrational numbers, yielding a similar result for the representing irrational numbers as fraction sequences. I encourage the reader to read more about this extension of the Euclidean algorithm in Niven, Zuckerman, and Montgomery's *An Introduction to the Theory of Numbers* [NZM].

As a consequence, we can write many irrational numbers as infinite fraction sequences. For example, for the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$, the Euclidean algorithm will tell us that $\phi = [\bar{1}] = [1, 1, 1, \ldots]$. This representation also suggests a useful way of approximating irrational numbers, i.e. by computing a finite portion of the infinite fraction sequence

$$\phi \approx [1, 1, 1, 1] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} \approx 1.667.$$

We can compute as many terms of the fraction sequence we would like to find better and better approximations. So the next time your middle school algebra teachers tell you that you cannot represent irrational numbers with fractions, you had better tell them otherwise!

## References

[NZM] Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery: *An Introduction to the Theory of Numbers*, 5th ed. Wiley, 1991.