FEATURE

11 Problems

The HCMR welcomes submissions of original problems in any fields of mathematics, as well as solutions to previously proposed problems. Proposers should direct problems to hcmr-problems@hcs.harvard.edu or to the address on the inside front cover. A complete solution or a detailed sketch of the solution should be included, if known. Solutions to previous problems should be directed to hcmr-solutions@hcs.harvard.edu or to the address on the inside front cover. Solutions should include the problem reference number, as well as the solver's name, contact information, and affiliated institution. Additional information, such as generalizations or relevant bibliographical references, is also welcome. Correct solutions will be acknowledged in future issues, and the most outstanding solutions received will be published. To be considered for publication, solutions to the problems below should be postmarked no later than *September 15, 2008*. An asterisk beside a problem or part of a problem indicates that no solution is currently available.

S08 – 1. It is known that there are 6670903752021072936960 square matrices M of order 9 with entries in $\{1, \ldots, 9\}$ that show valid sudoku grids.[1] How many of them have the property that the symmetric matrix $M + M^t$ is positive definite?

Proposed by Noam D. Elkies (Harvard University).

S08 – 2. Professor Perplex is at it again! This time, he has gathered his $n > 0$ combinatorial electrical engineering students and proposed:

> "I have prepared a collection of $r > 0$ identical rooms, each of which is empty except for $s > 0$ switches. You will be let into the rooms at random, in such a fashion that no two students occupy the same room at the same time and every student will visit each room arbitrarily many times. Once one of you is inside a room, he or she may toggle any of the s switches before leaving. This process will continue until some student chooses to assert that all the students have visited all the rooms at least $v > 0$ times each. If this student is right, then there will be no final exam this semester. Otherwise, I will assign a week-long final exam on the history of the light switch."

What is the minimal value of s (as a function of n, r, and v) for which the students can guarantee that they will not have to take an exam?

Proposed by Scott D. Kominers '09, Paul Kominers (Walt Whitman HS '08), and Justin Chen (Caltech '09).

S08 – 3. Let $k \geq 1$ be a natural number. Find all integer solutions to the diophantine equation

$$x^{2k+1} + x^{2k} + \cdots + x^2 + x + 1 = y^{2k+1}.$$

Proposed by Ovidiu Furdui (University of Toledo).

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[1] The proposer points out that this calculation is detailed in Bertram Felgenhauer and Frazer Jarvis: Enumerating possible Sudoku grids (2005), http://www.afjarvis.staff.shef.ac.uk/sudoku/sudoku.pdf, athough it was independently computed by user "QSCGZ" on the rec.puzzle Google group, thread "combinatorial question on 9x9," 21 Sep. 2003.

S08 – 4. Consider a, b, c three arbitrary positive real numbers. Prove that

$$\sum_{cyc}\sqrt{\frac{b+c}{a}} \geq 2\left(\sum_{cyc}\sqrt{\frac{a}{b+c}}\right)\cdot\sqrt{1+\frac{(a+b)(b+c)(c+a)-8abc}{4\sum_{cyc}a(a+b)(a+c)}}.$$

Proposed by Cosmin Pohoata (Bucharest, Romania).

S08 – 5. Let ABC be a non-isosceles triangle with $\angle A = 60^\circ$. Let H be its orthocenter and I its incenter. Let B_i and C_i the points such that the equilateral triangles ABC_i and AB_iC intersect the interior of ABC. Define B_e and C_e similarly, so that ABC_e and AB_eC are equilateral and disjoint from the interior of ABC.

Show that the lines through HI, B_iC_i and B_eC_e do not concur, and that the triangle they form is isosceles.

Proposed by Daniel Campos Salas (Costa Rica).

The following problem from the Fall 2007 issue received no submissions. Since this problem defied solution, we are rereleasing it for one more issue.

F07 – 5. For $i = 1, \ldots, n$, let $f_i : (\mathbb{Z}/m\mathbb{Z} \cup \{\star\})^n \to (\mathbb{Z}/m\mathbb{Z} \cup \{\star\})^n$ be given by

$$f_i\left((x_1,\ldots,x_n)\right) = \begin{cases} (\star, x_2+1, x_3, \ldots, x_n) & i = 1 \text{ and } x_1 = 1, \\ (x_1, \ldots, x_{i-1}+1, \star, x_{i+1}+1, \ldots, x_n) & 1 < i < n \text{ and } x_i = 1, \\ (x_1, \ldots, x_{n-2}, x_{n-1}+1, \star) & i = n \text{ and } x_n = 1, \\ (x_1, \ldots, x_n) & \text{otherwise,} \end{cases}$$

where $\star + 1 = \star$. Find necessary and sufficient conditions on $(x_1, \ldots, x_n) \in (\mathbb{Z}/m\mathbb{Z})^n$ such that there exists a sequence $\{i_k\}_{k=1}^n$ for which

$$f_{i_n}(\cdots(f_{i_1}((x_1,\ldots,x_n)))) = (\star,\ldots,\star).$$

Proposed by Paul Kominers (Walt Whitman HS '08), Scott D. Kominers '09, and Zachary Abel '10.